Exercise 28

For each of the following functions (i) give a definition like those in (2), (ii) sketch the graph, and (iii) find a formula similar to Equation 3.

(a) csch^{-1} (b) sech^{-1} (c) $\coth^{-1}$

Solution

Part (a)

The hyperbolic cosecant function is one-to-one because it satisfies the horizontal line test (see the graph in Exercise 22), so an inverse function exists.

$$y = \operatorname{csch}^{-1} x \quad \Leftrightarrow \quad \operatorname{csch} y = x$$

Express this equation on the right in terms of exponential functions.

$$\begin{aligned} x = \operatorname{csch} y = \frac{1}{\sinh y} = \frac{1}{\frac{e^y - e^{-y}}{2}} &= \frac{2}{e^y - e^{-y}} \\ &= \frac{2}{e^y - e^{-y}} \times \frac{e^y}{e^y} \\ &= \frac{2e^y}{e^{2y} - 1} \end{aligned}$$

Multiply both sides by $e^{2y} - 1$.

$$x(e^{2y} - 1) = 2e^y$$

$$xe^{2y} - x = 2e^y$$

$$xe^{2y} - 2e^y - x = 0$$

Use the quadratic formula to solve for e^y.

$$\begin{aligned} e^y &= \frac{2 \pm \sqrt{4 - 4(x)(-x)}}{2x} \\ &= \frac{2 \pm \sqrt{4 + 4x^2}}{2x} \\ &= \frac{2 \pm 2\sqrt{1 + x^2}}{2x} \\ &= \frac{1 \pm \sqrt{1 + x^2}}{x} \end{aligned}$$

The exponential function is always positive; choose the minus sign if x is negative, and choose the plus sign if x is positive.

$$e^y = \begin{cases} \dfrac{1-\sqrt{1+x^2}}{x} & \text{if } x < 0 \\[2ex] \dfrac{1+\sqrt{1+x^2}}{x} & \text{if } x > 0 \end{cases}$$

$$= \begin{cases} \dfrac{\sqrt{1+x^2}-1}{-x} & \text{if } x < 0 \\[2ex] \dfrac{\sqrt{1+x^2}+1}{x} & \text{if } x > 0 \end{cases}$$

$$= \frac{\sqrt{1+x^2}+\operatorname{sgn} x}{|x|}$$

Take the natural logarithm of both sides to solve for y.

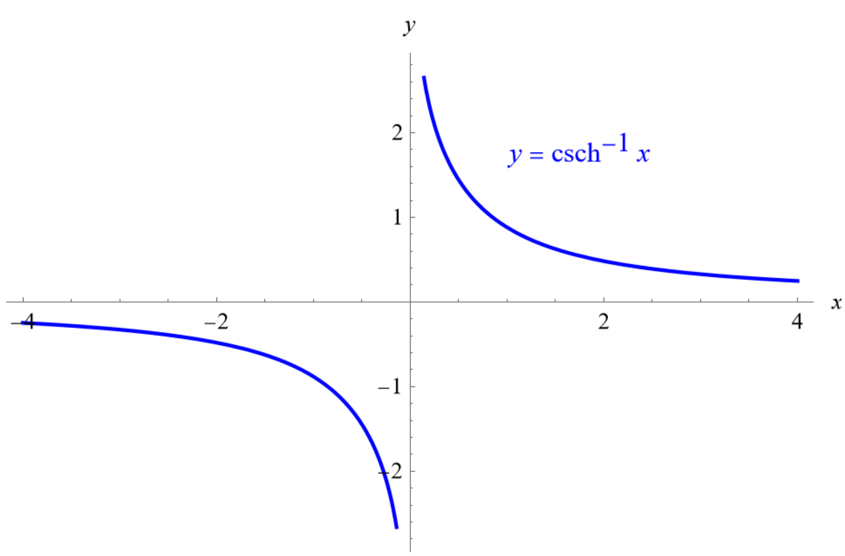
$$\ln e^y = \ln\left(\frac{\sqrt{1+x^2}+\operatorname{sgn} x}{|x|}\right)$$

$$y = \ln\left(\frac{\sqrt{1+x^2}+\operatorname{sgn} x}{|x|}\right)$$

Therefore,

$$\boxed{\operatorname{csch}^{-1} x = \ln\left(\frac{\sqrt{1+x^2}+\operatorname{sgn} x}{|x|}\right).}$$

A graph of this function is shown below.



Part (b)

The hyperbolic secant function is not one-to-one because it fails the horizontal line test (see the graph in Exercise 22), so an inverse function exists only if the restriction to nonnegative arguments is taken.

$$y = \operatorname{sech}^{-1} x \quad \Leftrightarrow \quad \operatorname{sech} y = x, \quad y \geq 0$$

Express this equation on the right in terms of exponential functions.

$$\begin{aligned} x = \operatorname{sech} y = \frac{1}{\cosh y} = \frac{1}{\frac{e^y + e^{-y}}{2}} &= \frac{2}{e^y + e^{-y}} \\ &= \frac{2}{e^y + e^{-y}} \times \frac{e^y}{e^y} \\ &= \frac{2e^y}{e^{2y} + 1} \end{aligned}$$

Multiply both sides by $e^{2y} + 1$.

$$x(e^{2y} + 1) = 2e^y$$

$$xe^{2y} + x = 2e^y$$

$$xe^{2y} - 2e^y + x = 0$$

Use the quadratic formula to solve for e^y.

$$\begin{aligned} e^y &= \frac{2 \pm \sqrt{4 - 4(x)(x)}}{2x} \\ &= \frac{2 \pm \sqrt{4 - 4x^2}}{2x} \\ &= \frac{2 \pm 2\sqrt{1 - x^2}}{2x} \\ &= \frac{1 \pm \sqrt{1 - x^2}}{x} \end{aligned}$$

Since $y \geq 0$, $e^y \geq 1$, which means the numerator has to be greater than or equal to the denominator. Choose the plus sign to make it so.

$$e^y = \frac{1 + \sqrt{1 - x^2}}{x}$$

Take the natural logarithm of both sides.

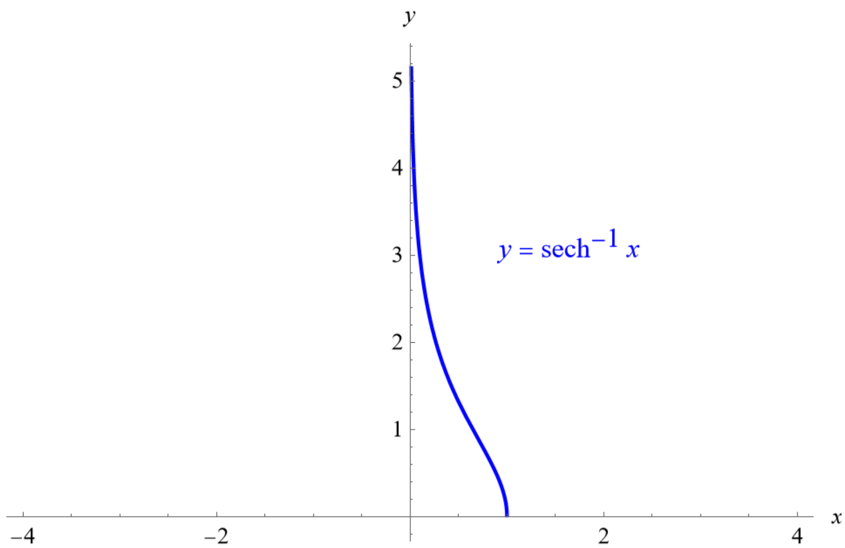
$$\ln e^y = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$$

$$y = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$$

Therefore,

$$\boxed{\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).}$$

A graph of this function is shown below.



Part (c)

The hyperbolic cotangent function is one-to-one because it satisfies the horizontal line test (see the graph in Exercise 22), so an inverse function exists.

$$y = \coth^{-1} x \quad \Leftrightarrow \quad \coth y = x$$

Express this equation on the right in terms of exponential functions.

$$\begin{aligned} x = \coth y = \frac{\cosh y}{\sinh y} = \frac{\frac{e^y+e^{-y}}{2}}{\frac{e^y-e^{-y}}{2}} &= \frac{e^y+e^{-y}}{e^y-e^{-y}} \\ &= \frac{e^y+e^{-y}}{e^y-e^{-y}} \times \frac{e^y}{e^y} \\ &= \frac{e^{2y}+1}{e^{2y}-1} \end{aligned}$$

Multiply both sides by $e^{2y}-1$.

$$x(e^{2y}-1) = e^{2y}+1$$

$$xe^{2y} - x = e^{2y}+1$$

$$xe^{2y} - e^{2y} = x+1$$

Factor the exponential function and then solve for e^{2y}.

$$(x-1)e^{2y} = x+1$$

$$e^{2y} = \frac{x+1}{x-1}$$

Take the natural logarithm of both sides to solve for y.

$$\ln e^{2y} = \ln\left(\frac{x+1}{x-1}\right)$$

$$2y \ln e = \ln\left(\frac{x+1}{x-1}\right)$$

$$y = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

Therefore,

$$\boxed{\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right).}$$